REAL AND COMPLEX VARIABLES PH.D. QUALIFYING EXAM

January 10, 2019

Three Hours

A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let $X \subseteq Y \subseteq Z$ where Z is a metric space (hence X and Y are metric subspaces). Prove that if X is dense in Y and Y is dense in Z, then X is dense in Z. (You may use any of the equivalent definitions of "dense" to do this, but state clearly which one(s) you are invoking.)

2. Explain why the function $f(x) = \sum_{n=1}^{\infty} \dfrac{x^n \sin(x^n)}{n!}$ is well-defined and continuous on all of $\mathbb{R}$.

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine whether or not f is Riemann integrable on [0,1]; and if so, find its Riemann integral. (You may use either Riemann's Condition or Lebesgue's Theorem.)

   (b) Explain briefly why f is Lebesgue integrable, and find its Lebesgue integral $\int_0^1 f$.

4. Let A be a subset of $[0, 1]$ that has *outer* Lebesgue measure a.

   (a) Prove that there is some Lebesgue measuable set B such that $A \subseteq B$ and B has Lebesgue measure a.

   (b) Explain why $B - A$ need *not* have measure zero.

5. Find, with justification, the value of

$$\lim_{N\to\infty} \int_0^1 N \sin\left(\frac{x^2}{N}\right)\, dx.$$

6. Let ℓ^2 be the usual Hilbert space of square summable sequences of real numbers. Prove that the closed unit ball, $B = \{\, \{a_n\}_{n=1}^{\infty} \mid \sum a_n^2 \leq 1\}$, is not compact.

Section B. Complex Analysis

7. Let f be an analytic function defined on the open unit disc $\mathbb{D} = \{z \mid |\, z\,| < 1\}$. Prove that if f is real valued on $\mathbb{D}$, then it must be constant.

8. Prove that $\log z = e^z$ has no solution for $|\, z\,| = 1$, where $\log z$ denotes the principal branch of the logarithm.

9. Use the method of residues to find the value of the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+3)^2}\, dx$.
(Sketch your contour of integration and very briefly justify your method.)

10. Find the Laurent series of the form $\sum\limits_{n=-\infty}^{\infty} c_n z^n$ for

$$f(z) = \frac{1}{(z+1)^2} + \frac{z}{z^2+8}$$

that converges in an annulus containing the point $z = 2i$, and state precisely where this Laurent series converges.

11. Let $\mathbb{D} = \{z \mid |\, z\,| < 1\}$ be the open unit disc and let $\mathbb{D}^* = \mathbb{D} - \{0\}$. Prove that there is no one-to-one, onto analytic map $f : \mathbb{D}^* \to \mathbb{D}$.

12. Let $f(z)$ be an entire function and let $p(z)$ be a non-constant polynomial. Prove that if

$$|\, f(z)\,| \le |\, f(z) + p(z)\,|, \qquad \text{for all } z \in \mathbb{C}$$

then $f(z) = kp(z)$ for some $k \in \mathbb{C}$.

13. Find with justification a positive integer n such that $p(z) = z^5 + nz^2 + \pi z + 1$ has exactly two zeros in the open disc of radius 2 centered at the origin.