COMPLEX VARIABLES PH.D. QUALIFYING EXAM

September 27, 2008

There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let $f(z) = z|\,z\,|^2$.

(a) Find all points in the complex plane where f satisfies the Cauchy-Riemann equations.

(b) Does f have a complex derivative at the points you found in (a)? (Justify briefly.)

2. Find all complex numbers z such that $\tan z = i - 1$.

3. Let $f(z) = \overline{z}$. In each of parts (a) and (b) compute $\int_\gamma f(z)\,dz$, where γ is the specified path whose initial point is -1 and terminal point is i.

(a) γ is the path along the coordinate axes: from -1 to 0 and then from 0 to i.

(b) γ is the quarter of the unit circle lying in the second quadrant, oriented clockwise.

(c) Could there be a function g that is analytic on some simply connected open set U containing both the paths in (a) and (b) such that $g = f$ on both paths? (Explain briefly.)

4. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point $z = 3$ for the following function (explain briefly how the inner and outer radii of the annulus are determined):

$$f(z) = \frac{z}{z^2 - 4} + \frac{12}{(z-4)^2}.$$

5. Use the calculus of residues to evaluate the improper integral

$$\int_{1}^{1} \frac{x \sin x}{x^2 + 9}\,dx.$$

[Draw your path of integration; but you may quote without proof any standard estimates for integrals along portions of your path, making sure to mention what growth conditions are required.]

6. Suppose f is entire and there is a positive real number M and a polynomial p such that $|\,f(z)\,| \le |\,p(z)\,|$ for all z with $|\,z\,| > M$. Prove that f is a polynomial.

7. Let f and g be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume both f and g have no zeros in $\overline{\mathbb{D}}$. Prove that if $|\,f(z)\,| = |\,g(z)\,|$ for all z with $|\,z\,| = 1$, then $f(z) = kg(z)$ in $\mathbb{D}$ for some constant k of modulus 1.

8. (a) Exhibit a function f such that at each positive integer n, f has a pole of order n, and f is analytic and nonzero at every other complex number. (Briefly justify your answer.)

(b) Let f be any function that satisfies the conditions of part (a). For each positive integer N find $\displaystyle\int_{C_N} \frac{f'(z)}{f(z)}\,dz$, where C_N is the circle of radius $N + \frac{1}{2}$ centered at the origin.

9. (a) State Goursat's Theorem.

(b) Use Goursat's Theorem to prove that if f is continuous on $\mathbb{C}$ and analytic at every point not on the real axis, then f must be analytic everywhere.

10. Suppose f is entire and for some positive real number K

$$|\,\mathrm{Re}\ f(z)\,| \geq |\,\mathrm{Im}\ f(z)\,|, \qquad \text{for all } z \text{ with } |\,z\,| \geq K.$$

Prove that f is constant on $\mathbb{C}$.